Every compact group arises as the outer automorphism group of a II$_1$ factor

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Abstract

We show that any compact group can be realized as the outer automorphism group of a factor of type II$_1$. This has been proved in the abelian case by Ioana, Peterson and Popa [6] applying Popa's deformation/rigidity techniques to amalgamated free product von Neumann algebras. Our methods are a generalization of theirs.

Introduction

The outer automorphism group $\operatorname{Out}(M)$ of a II$_1$ factor M provides, in principle, a useful invariant to distinguish between families of II$_1$ factors. But this group $\operatorname{Out}(M)$ is extremely hard to compute.

Breakthrough rigidity results in the theory of II$_1$ factors were obtained recently by Popa (see [10, 11, 12]) and are based on Popa's deformation/rigidity technique. Techniques and results of Popa were applied in several papers (see also [15] for an overview) and two papers included as an application complete computations of outer automorphism groups of certain II$_1$ factors.

- In [6], it is shown that there exists, for every *compact abelian* group K, a type II$_1$ factor M with $\operatorname{Out}(M) \cong K$. The result in [6] is an existence theorem and answered in particular the long standing open problem on the existence of II$_1$ factors without outer automorphisms.

- In [14], type II$_1$ factors M with $\operatorname{Out}(M)$ any *discrete group of finite presentation* are explicitly constructed. This in particular gave the first explicit examples of II$_1$ factors without outer automorphisms.

In our paper, the methods of [6] are applied to prove the existence of II$_1$ factors M such that $\operatorname{Out}(M)$ is any, possibly non-abelian, compact group. In fact, given a minimal action of a compact group G on the hyperfinite II$_1$ factor R, we prove the existence of an action of $\Gamma = \operatorname{SL}(3, \mathbb{Z})$ on the fixed point algebra R^G such that the II$_1$ factor M given as the amalgamated free product $M = (R^G \rtimes \Gamma) *_{R^G} R$ satisfies $\operatorname{Out}(M) \cong G$.

The first rigidity results for II$_1$ factors are due to Connes in [1], where it is shown in particular that $\operatorname{Out}(N)$ is a countable group whenever $N = \mathcal{L}(\Gamma)$ is the group von Neumann algebra of an ICC property (T) group Γ. So, for concrete ICC property (T) groups Γ, the group $\operatorname{Out}(\mathcal{L}(\Gamma))$ is in principle computable, although we do not know of any explicit computation.

Type II$_1$ factors admit a more general type of symmetry, under the form of *finite index bimodules*. The finite index M-M-bimodules (modulo isomorphism) form a *fusion algebra* that we denote as $\operatorname{FAlg}(M)$. Such a fusion algebra is a set equipped with an additive (direct sum) and a multiplicative

(tensor product) structure and in which every element is the finite direct sum of irreducible elements. Another generalization of [6] was given by the second author in [16], where the existence of II_1 factors M with trivial fusion algebra, was shown. In our paper the fusion algebra $\mathrm{FAlg}(R)$ of the hyperfinite II_1 factor plays an important role: we make use of the fact that two countable fusion subalgebras of $\mathrm{FAlg}(R)$ become free after conjugating one of them by a well chosen automorphism of R (see [16]).

1 Preliminaries

We denote by (M, τ) a von Neumann algebra M equipped with a faithful normal tracial state τ. We denote $M^n := M_n(\mathbb{C}) \otimes M$ for all $n \in \mathbb{N}$.

Let (M, τ) be a tracial von Neumann algebra and $N \subset M$ a von Neumann subalgebra. The *-algebra of elements quasi-normalizing N is defined as

$$\mathrm{QN}_M(N) := \{a \in M \mid \exists a_1, \ldots, a_n, b_1, \ldots, b_m \in M \text{ such that } Na \subset \sum_{i=1}^n a_i N \text{ and } aN \subset \sum_{i=1}^m N b_i\}$$

The inclusion $N \subset M$ is called *quasi-regular* if $\mathrm{QN}_M(N)'' = M$.

If N is a von Neumann subalgebra of a von Neumann algebra M, we denote by $\mathrm{Aut}(N \subset M)$ the group of automorphisms of M leaving N globally invariant.

1.1 Amalgamated free products

We make use of amalgamated free product factors and recall some basic facts and notations (see [9] and [17] for more details). Let (M_0, τ_0) and (M_1, τ_1) be tracial von Neumann algebras with a common von Neumann subalgebra N such that $\tau_{0|N} = \tau_{1|N}$. We denote by E_i the unique τ-preserving conditional expectation of M_i onto N. The amalgamated free product $M_0 *_N M_1$ is, up to E-preserving isomorphism, the unique pair (M, E) satisfying the following two conditions.

- The von Neumann algebra M is generated by embeddings of M_0 and M_1 that are identical on N, and is equipped with a conditional expectation $E : M \to N$.

- The subalgebras M_0 and M_1 are free with amalgamation over N with respect to E. This means that $E(x_1 \cdots x_n) = 0$ whenever $x_j \in M_{i_j}$ such that $E_{i_j}(x_j) = 0$ and $i_1 \neq i_2$, $i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$.

The amalgamated free product $M_0 *_N M_1$ has a dense *-subalgebra given by:

$$N \;\oplus\; \bigoplus_{n \geq 1} \left(\bigoplus_{i_1 \neq i_2, \ldots, i_{n-1} \neq i_n} \overset{\circ}{M_{i_1}} \cdots \overset{\circ}{M_{i_n}} \right)$$

where $\overset{\circ}{M_{i_k}} := M_{i_k} \ominus N$. The von Neumann algebra $M_0 *_N M_1$ has a trace, defined by $\tau := \tau_0 \circ E = \tau_1 \circ E$.

1.2 Popa's intertwining-by-bimodules technique

In this paper, we use Popa's intertwining-by-bimodules technique (see [10, 11, 12]) that we briefly recall now. Let (B, τ) be a tracial von Neumann algebra and H_B a right Hilbert B-module. There exists a projection $p \in \mathrm{B}(\ell^2(\mathbb{N})) \otimes B$ such that $H_B \cong p(\ell^2(\mathbb{N}) \otimes L^2(B, \tau))_B$ and this projection is uniquely defined up to equivalence of projections in $\mathrm{B}(\ell^2(\mathbb{N})) \otimes B$. We denote $\dim(H_B) := (\mathrm{Tr} \otimes \tau)(p)$. Observe that the number $\dim(H_B)$ depends on the choice of tracial state τ in the non-factorial case.

Suppose now that A and B are two possibly non-unital von Neumann subalgebras of a tracial von Neumann algebra (N, τ). We say that A embeds into B inside N, and write $A \prec_N B$, if there exists a non-zero A-B-subbimodule H of $1_A L^2(N) 1_B$ such that $\dim(H_B) < +\infty$. The relation $A \prec_N B$ is independent of the choice of tracial state τ and is equivalent with the existence of $n \in \mathbb{N}$, $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes 1_A N 1_B$ a non-zero partial isometry and $\psi : A \to \mathrm{M}_n(\mathbb{C}) \otimes B$ a possibly non-unital *-homomorphism satisfying $av = v\psi(a)$ for all $a \in A$. For details, we refer to Section 2 in [10] and Appendix C in [15].

1.3 Fusion algebras

We first recall the abstract notion of a *fusion algebra* and give below the basic example of the fusion algebra $\mathrm{FAlg}(M)$ of finite index bimodules over the II_1 factor M.

Definition 1.1. *A fusion algebra $\mathcal{A}$ is a free $\mathbb{N}$-module $\mathbb{N}[\mathcal{G}]$ equipped with the following additional structure:*

- *an associative and distributive product operation, and a multiplicative unit element $e \in \mathcal{G}$,*

- *an additive, anti-multiplicative, involutive map $x \mapsto \overline{x}$, called* conjugation,

satisfying Frobenius reciprocity: defining the numbers $m(x, y; z) \in \mathbb{N}$ for $x, y, z \in \mathcal{G}$ through the formula

$$xy = \sum_z m(x, y; z) z$$

one has $m(x, y; z) = m(\overline{x}, z; y)$ for all $x, y, z \in \mathcal{G}$.

The base $\mathcal{G}$ of the fusion algebra $\mathcal{A}$ is canonically determined: these are exactly the non-zero elements of $\mathcal{A}$ that cannot be expressed as the sum of two non-zero elements. The elements of $\mathcal{G}$ are called the *irreducible elements* of the fusion algebra $\mathcal{A}$ and we sometimes write $\mathcal{G} = \mathrm{Irred}\,\mathcal{A}$. Notice that conjugation preserves irreducibility. The *intrinsic group* $\mathrm{grp}(\mathcal{A})$ of the fusion algebra $\mathcal{A}$ consists of the irreducible elements $x \in \mathcal{A}$ such that $x\overline{x} = e$. Equivalently, $x \in \mathcal{A}$ belongs to the intrinsic group if and only if $x\overline{x}$ is irreducible. It is easy to check that the intrinsic group of a fusion algebra is indeed a group. If $x, y \in \mathcal{A}$, we sometimes say that x *is included in* y, if there exists a z such that $y = x + z$.

A *dimension function* on a fusion algebra $\mathcal{A}$ is an additive, multiplicative, unital map $d : \mathcal{A} \to \mathbb{R}^+$ satisfying $d(\overline{x}) = d(x)$ for all $x \in \mathcal{A}$. Suppose d is a dimension function on the fusion algebra $\mathcal{A}$. Whenever $x \in \mathcal{A}$ is non-zero, e is included in $x\overline{x}$ and so $d(x) \geq 1$. It then follows that $x \in \mathcal{A}$ belongs to the intrinsic group of $\mathcal{A}$ if and only if $d(x) = 1$. Moreover, if $x \in \mathcal{A}$ is non-zero and not in the intrinsic group, the same reasoning yields $d(x) \geq \sqrt{2}$.

Two examples of fusion algebras with a dimension function arise as follows.

- Let Γ be a group and define $\mathcal{A} = \mathbb{N}[\Gamma]$. Define d such that $d(s) = 1$ for all $s \in \Gamma$.

- Let G be a compact group and define the fusion algebra $\mathrm{Rep}(G)$ as the set of equivalence classes of finite dimensional unitary representations of G. The operations on $\mathrm{Rep}(G)$ are of course given by direct sum and tensor product of representations, while the dimension function d is given by the ordinary Hilbert space dimension of the representation space.

We are interested in the fusion algebra $\mathrm{FAlg}(M)$ of a II_1 factor M. First of all, an M-M-bimodule $_M H_M$ is said to be of *finite Jones index* if $\dim(_M H) < \infty$ and $\dim(H_M) < \infty$. We define $\mathrm{FAlg}(M)$ as the set of finite index M-M-bimodules modulo unitary equivalence.

Whenever $\psi : M \to pM^n p$ is a finite index inclusion in the sense of Jones [7], for some non-zero projection $p \in M^n$, we define the M-M-bimodule $H(\psi)$ on the Hilbert space $(\mathrm{M}_{1,n}(\mathbb{C}) \otimes L^2(M))p$ with left and right module actions given by

$$a \cdot \xi := a\xi \quad \text{and} \quad \xi \cdot a = \xi\psi(a) .$$

Every finite index M-M-bimodule is unitarily equivalent with some $H(\psi)$. Moreover, given finite index inclusions $\psi : M \to pM^n p$ and $\eta : M \to qM^m q$, we have $H(\psi) \cong H(\eta)$ if and only if there exists a unitary $u \in p(\mathrm{M}_{n,m}(\mathbb{C}) \otimes M)q$ satisfying $\psi(a) = u\eta(a)u^*$ for all $a \in M$.

Addition in $\mathrm{FAlg}(M)$ is given by the obvious direct sum of bimodules, while multiplication in $\mathrm{FAlg}(M)$ is given by the *Connes tensor product of M-M-bimodules* that we recall now (see also V.Appendix B in [2]). Let H be an M-M-bimodule. Define $\mathcal{H}$ as the dense subspace of H consisting of bounded vectors:

$$\mathcal{H} := \{\xi \in H \mid \exists c > 0, \ \forall a \in M, \ \|\xi a\|_2 \le c\|a\|_2\} .$$

For all $\xi \in \mathcal{H}$ and $a \in M$ we define $L_\xi(a) = \xi a$. By definition this map extends to a bounded operator $L_\xi : L^2(M) \to H$. We set:

$$\langle \xi, \eta \rangle_M := L_\xi^* L_\eta \in M, \ \forall \xi, \eta \in \mathcal{H} .$$

It is easy to check that this formula defines an M-valued scalar product on $\mathcal{H}$. Then the Connes tensor product of the M-M-bimodules $_M H_M$ and $_M K_M$ is defined as the separation and completion of the algebraic tensor product $\mathcal{H} \otimes_{\mathrm{alg}} K$ for the scalar product

$$\langle a \otimes \xi, b \otimes \eta \rangle := \langle \xi, \langle a, b \rangle_M \eta \rangle .$$

The Connes tensor product is denoted by $H \otimes_M K$ and it is an M-M-bimodule:

$$a \cdot (b \otimes \xi) = ab \otimes \xi \quad \text{and} \quad (b \otimes \xi) \cdot a = b \otimes (\xi a) .$$

Note that $H(\psi) \otimes_M H(\eta) = H((\mathrm{id} \otimes \psi)\eta)$.

If $_M H_M \in \mathrm{FAlg}(M)$, the *conjugate bimodule* $_M \overline{H}_M$ lives on the conjugate Hilbert space $\overline{H} = H^*$ with bimodule action given by

$$a \cdot \overline{\xi} = \overline{\xi a^*} \quad \text{and} \quad \overline{\xi} \cdot a = \overline{a^* \xi} .$$

The elements of the *intrinsic group* $\mathrm{grp}(M)$ of $\mathrm{FAlg}(M)$ are exactly the bimodules $H(\pi)$, where $\pi : M \to pM^n p$ is a *-isomorphism. Denote by $\mathcal{F}(M)$ the *fundamental group* of M. We then get a short exact sequence $e \to \mathrm{Out}(M) \to \mathrm{grp}(M) \to \mathcal{F}(M) \to e$, mapping $\sigma \in \mathrm{Aut}(M)$ to $H(\sigma) \in \mathrm{grp}(M)$ and mapping $H(\pi) \in \mathrm{grp}(M)$ to $\mathrm{Tr}(p)$.

The fusion algebra $\mathrm{FAlg}(M)$ has a natural dimension function: the dimension of $H(\psi)$ is defined as the square root of the *minimal index* of $\psi(M) \subset pM^np$. Since for an irreducible subfactor the minimal index equals the usual Jones index, the dimension function d is given by

$$d(_M H_M) = \sqrt{\dim(H_M)\ \dim(_M H)}\,,$$

whenever $_M H_M$ is an *irreducible M-M*-bimodule. We refer to [8, 4] for details.

1.4 Minimal actions of compact groups and fusion algebras

A continuous action $G \curvearrowright M$ of a compact group G on the II_1 factor M is said to be *minimal* if the map $G \to \mathrm{Aut}(M)$ is injective and if $M \cap (M^G)' = \mathbb{C}1$. Here, M^G denotes the von Neumann algebra of G-fixed points in M.

Given such a minimal action $G \curvearrowright M$, set $N := M^G$. We get a canonical, dimension preserving, embedding $\mathrm{Rep}(G) \hookrightarrow \mathrm{FAlg}(N)$ of fusion algebras, defined as follows. Let $\pi : G \to \mathcal{U}(n)$ be an irreducible unitary representation of G. We choose a unitary $V_\pi \in \mathrm{M}_n(\mathbb{C}) \otimes M$ satisfying

$$(\mathrm{id} \otimes \sigma_g)(V_\pi) = V_\pi(\pi(g) \otimes 1)$$

for all $g \in G$. We then define the finite index inclusion

$$\psi_\pi : N \to \mathrm{M}_n(\mathbb{C}) \otimes N : \psi_\pi(a) = V_\pi(1 \otimes a)V_\pi^* \,.$$

It is easily checked that the *N-N*-bimodule $H(\psi_\pi)$ is irreducible and, up to unitary equivalence, independent of the choice of V_π. The map $\pi \mapsto H(\psi_\pi)$ extends to an embedding $\mathrm{Rep}(G) \hookrightarrow \mathrm{FAlg}(N)$.

Also note that the coefficients of V_π quasi-normalize N and so, the inclusion $N \subset M$ is *quasi-regular*.

1.5 Freeness and free products of fusion algebras

Definition 1.2. *Let $\mathcal{A}$ be a fusion algebra and $\mathcal{A}_i \subset \mathcal{A}$ fusion subalgebras for $i = 1, 2$. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are* free inside $\mathcal{A}$ *if every alternating product of irreducibles in $\mathcal{A}_i \setminus \{e\}$, remains irreducible and different from $\{e\}$.*

Given fusion algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, there is up to isomorphism a unique fusion algebra $\mathcal{A}$ generated by copies of $\mathcal{A}_1$ and $\mathcal{A}_2$ that are free. We call this unique $\mathcal{A}$ the *free product* of $\mathcal{A}_1$ and $\mathcal{A}_2$ and denote it by $\mathcal{A}_1 * \mathcal{A}_2$. Of course, the free product can be constructed in a concrete way as follows: given $\mathcal{A}_1$ and $\mathcal{A}_2$, set $\mathcal{G}_i = \mathrm{Irred}(\mathcal{A}_i)$. Define $\mathcal{G}$ as the set of words with letters alternatingly from $\mathcal{G}_1 \setminus \{e\}$ and $\mathcal{G}_2 \setminus \{e\}$. Denote the empty word as e. Then, $\mathcal{A}_1 * \mathcal{A}_2 = \mathbb{N}[\mathcal{G}]$. The product on $\mathbb{N}[\mathcal{G}]$ is the unique associative and distributive operation satisfying the following two conditions:

- The embeddings $\mathcal{A}_i \hookrightarrow \mathbb{N}[\mathcal{G}]$ are multiplicative.

- If the last letter of the alternating word $x \in \mathcal{G}$ and the first letter of the alternating word $y \in \mathcal{G}$ belong to different fusion algebras $\mathcal{A}_i$, the product of x and y is again irreducible and given by concatenation of x and y.

Denote by R the hyperfinite II_1 factor. It is a crucial ingredient of this paper that $\mathrm{FAlg}(R)$ is huge, in the sense that $\mathrm{FAlg}(R)$ contains many free fusion subalgebras. More precisely, it was shown in Theorem 5.1 of [16] that countable fusion subalgebras of $\mathrm{FAlg}(R)$ can be made free by conjugating one of them with an automorphism of R (see Theorem 1.3 below). Note that the same result has first been proven for countable subgroups of $\mathrm{Out}(R)$ in [6]. In both cases, the key ingredients come from [13].

Let M be a II_1 factor and $_M K_M \in \mathrm{FAlg}(M)$. Whenever $\alpha \in \mathrm{Aut}(M)$, we define the conjugation of K by α as the bimodule $K^\alpha := H(\alpha^{-1}) \otimes_M K \otimes_M H(\alpha)$. Of course, K^α has K as its underlying Hilbert space with new left and right module action given by $\xi \cdot_{\mathrm{new}} a = \xi \cdot_{\mathrm{old}} \alpha(a)$ and $a \cdot_{\mathrm{new}} \xi = \alpha(a) \cdot_{\mathrm{old}} \xi$.

Theorem 1.3 (Thm. 5.1 in [16]). *Let R be the hyperfinite II_1 factor and $\mathcal{A}_0, \mathcal{A}_1$ two countable fusion subalgebras of $\mathrm{FAlg}(R)$. Then,*

$$\{\alpha \in \mathrm{Aut}(R) \mid \mathcal{A}_0^\alpha \text{ and } \mathcal{A}_1 \text{ are free}\}$$

is a G_δ-dense subset of $\mathrm{Aut}(R)$.

1.6 Property (T) and relative property (T) for II_1 factors

Property (T) for finite von Neumann algebras was defined by Connes and Jones in [3]: a II_1 factor (N, τ) has property (T) if and only if there exists $\epsilon > 0$ and a finite subset $F \subset N$ such that every N-N-bimodule H that has a unit vector ξ satisfying $\|x\xi - \xi x\| \le \epsilon$, $\forall x \in F$, actually has a non-zero N-central vector ξ_0, meaning that $x\xi_0 = \xi_0 x$, $\forall x \in N$.

Note that an ICC group Γ has property (T) if and only if the II_1 factor $\mathcal{L}(\Gamma)$ has property (T) in the sense of Connes and Jones.

Relative property (T) for inclusions $B \subset (N, \tau)$ of finite von Neumann algebras was defined by Popa in [12]. We do not really use relative property (T) in this paper, just the trivial observation that $B \subset N$ has the relative property (T) if B has itself property (T).

2 Statement of the main result

The main result that we prove is that any compact group G can be realized as the outer automorphism group of a type II_1 factor, see Theorem 2.3. A more precise theorem can be formulated as follows.

Note that any character $\omega \in \mathrm{Char}(\Gamma)$ defines an automorphism θ_ω of any crossed product $N \rtimes \Gamma$ acting identically on N and multiplying by ω on Γ.

Theorem 2.1. *Let M_1 be the hyperfinite II_1 factor and G a compact group acting on M_1. Denote $N = M_1^G$, the von Neumann algebra of G-fixed points in M_1. Let Γ be an ICC group acting on N. Denote $M_0 := N \rtimes \Gamma$. Assume that*

1. *the action $G \curvearrowright M_1$ is minimal,*

2. *the action $\Gamma \curvearrowright N$ is outer and M_0 has the property (T),*

3. *the natural images of $\mathrm{Rep}\, G \hookrightarrow \mathrm{FAlg}(N)$ and $\mathrm{Aut}(N \subset M_0) \xrightarrow{\mathrm{restr}} \mathrm{Out}(N) \subset \mathrm{FAlg}(N)$ inside the fusion algebra $\mathrm{FAlg}(N)$, are free in the sense of Definition 1.2.*

Then, the homomorphism

$$\mathrm{Char}(\Gamma) \times G \to \mathrm{Aut}(M_0 \underset{N}{*} M_1) : (\omega, g) \mapsto \theta_\omega * \sigma_g$$

induces an isomorphism $\mathrm{Char}(\Gamma) \times G \cong \mathrm{Out}(M_0 \underset{N}{*} M_1)$.

Combining Theorems 1.3 and 2.1, we shall prove the following.

Corollary 2.2. *Let G be a compact, second countable group and $G \curvearrowright R$ a minimal action on the hyperfinite II_1 factor R. Then there exists an outer action of $\mathrm{SL}(3, \mathbb{Z})$ on the fixed point algebra R^G, such that for M given as the amalgamated free product $M = (R^G \rtimes \Gamma) \underset{R^G}{*} R$, the natural homomorphism*

$$G \to \mathrm{Aut}(M) : g \mapsto \mathrm{id} * \sigma_g$$

induces an isomorphism $G \cong \mathrm{Out}(M)$.

Of course, every compact, second countable group G admits a minimal action on the hyperfinite II_1 factor. A possible construction goes as follows: take an amenable ICC group Λ and define the Bernoulli action crossed product $R = L^\infty\big(\prod_\Lambda (G, \mathrm{Haar})\big) \rtimes \Lambda$ with $G \curvearrowright R$ acting by diagonal left translation on $\prod_\Lambda (G, \mathrm{Haar})$ and trivially on Λ.

So, we immediately get the following result.

Theorem 2.3. *Let G be a compact, second countable group. There exists a type II_1 factor M with $\mathrm{Out}(M) \cong G$.*

3 A Kurosh automorphism theorem for fusion algebras

An important ingredient in the analysis of all automorphisms of an amalgamated free product $M = M_0 *_N M_1$ as in Theorem 2.1 above, is a generalization of the Kurosh automorphism theorem to automorphisms of free products of fusion algebras. We do not prove a general result, but a rather easy theorem sufficient for our purposes.

Recall that a group is said to be freely indecomposable if it cannot be written as a non-trivial free product.

Theorem 3.1. *Let Γ be a countable group non isomorphic to $\mathbb{Z}$ and freely indecomposable. Let $\mathcal{A}$ be an abelian fusion algebra with a dimension function d and non-isomorphic to the group $\mathbb{Z}$.*

Every dimension preserving automorphism α of $\mathbb{N}[\Gamma] * \mathcal{A}$ is of the form $(\mathrm{Ad}\, u) \circ (\alpha_0 * \alpha_1)$ for some $u \in \Gamma * \mathrm{grp}(\mathcal{A})$, $\alpha_0 \in \mathrm{Aut}(\Gamma)$ and α_1 a dimension preserving automorphism of $\mathcal{A}$.

Proof. Let α be a dimension preserving automorphism of $\mathbb{N}[\Gamma] * \mathcal{A}$.

We denote $\Lambda = \mathrm{grp}(\mathcal{A})$, the intrinsic group of $\mathcal{A}$, and $\Delta = \Gamma * \Lambda$, which is as well the intrinsic group of $\mathbb{N}[\Gamma] * \mathcal{A}$. We also write $\mathcal{G} = \mathrm{Irred}\,\mathcal{A}$, which means that $\mathcal{A} = \mathbb{N}[\mathcal{G}]$. We may of course assume that $\mathcal{G} \neq \Lambda$, because the group case of our theorem is covered by the classical Kurosh theorem. Finally, we set $\mathcal{G}^\circ = \mathcal{G} \setminus \{e\}$ and $\Gamma^\circ = \Gamma \setminus \{e\}$. If $u \in \Delta$, we write u^{-1} instead of $\overline{u}$.

Claim. There exists $x \in \mathcal{G} \setminus \Lambda$ and $u \in \Delta$ such that $\alpha(x) \in u(\mathcal{G} \setminus \Lambda)u^{-1}$.

Proof of the claim. Define $\lambda = \inf\{d(x) \mid x \in \mathcal{G} \setminus \Lambda\} \geq \sqrt{2}$. Take $x \in \mathcal{G} \setminus \Lambda$ with $d(x) < \sqrt{2}\lambda$. Write $\alpha(x)$ as an alternating word in $\mathcal{G}^\circ$ and Γ°. Suppose that in this expression of $\alpha(x)$, there

appears twice a letter from $\mathcal{G} \setminus \Lambda$. Then the dimension of these two letters is greater or equal than $\lambda \geq \sqrt{2}$, making $d(x) = d(\alpha(x)) \geq \sqrt{2}\lambda$; a contradiction. So, we have shown that $\alpha(x) = uyv^{-1}$ with $y \in \mathcal{G} \setminus \Lambda$ and $u, v \in \Delta$. We may assume that u, v are either equal to e, either end with a letter from Γ°. Expressing the commutation of $\alpha(x)$ and $\alpha(\overline{x})$, we find that $uy\overline{y}u^{-1} = vy\overline{y}v^{-1}$. Since $y \notin \Lambda$, we find that $y\overline{y} \neq e$ and so $u = v$, proving the claim.

Observation 1. If $x \in \Delta$, $y \in \mathcal{G}^\circ$ and $xy = yx$, then $x \in \Lambda$. This follows by analyzing reduced words in Γ° and $\mathcal{G}^\circ$.

Because of the claim and replacing α by $(\operatorname{Ad} u^{-1}) \circ \alpha$, we may from now on assume the existence of $x, y \in \mathcal{G} \setminus \Lambda$ with $\alpha(x) = y$. Whenever $a \in \Lambda$, $\alpha(a)$ belongs to Δ and commutes with y. Observation 1 above implies that $\alpha(\Lambda) \subset \Lambda$. Similarly, $\alpha^{-1}(\Lambda) \subset \Lambda$ so that $\alpha(\Lambda) = \Lambda$. It follows that the restriction of α to Δ defines an automorphism of $\Gamma * \Lambda$ that globally preserves Λ. The classical Kurosh theorem implies that $\alpha(\Gamma) = \Gamma$.

Observation 2. If $z \in \mathcal{G}^\circ$ and $\alpha(z) \in \Delta\mathcal{G}^\circ\Delta$, then actually $\alpha(z) \in \mathcal{G}^\circ$. Indeed, write $\alpha(z) = urv^{-1}$ for $r \in \mathcal{G}^\circ$ and $u, v \in \Delta$ either equal to e or with their last letter in Γ°. Writing out that $\alpha(z) = urv^{-1}$ and $y = \alpha(x)$ commute, it follows that $u = v = e$. A similar observation holds for α^{-1}.

It remains to prove that $\alpha(\mathcal{G}) = \mathcal{G}$. Assume the contrary and define

$$\delta = \inf\{d(z) \mid z \in \mathcal{G}, \quad (\alpha(z) \notin \mathcal{G} \text{ or } \alpha^{-1}(z) \notin \mathcal{G}) \} .$$

Take $z \in \mathcal{G}^\circ$ with $d(z) < \sqrt{2}\delta$ such that $\alpha(z) \notin \mathcal{G}^\circ$ or $\alpha^{-1}(z) \notin \mathcal{G}^\circ$. Assume that we are in the case $\alpha(z) \notin \mathcal{G}^\circ$. By construction $\alpha(r), \alpha^{-1}(r) \in \mathcal{G}^\circ$ for every $r \in \mathcal{G}^\circ$ with $d(r) < \delta$. Write $\alpha(z)$ as an alternating word in $\mathcal{G}^\circ$ and Γ°. By observation 2, the expression for $\alpha(z)$ contains at least twice a letter from $\mathcal{G} \setminus \Lambda$. Hence every letter in the expression for $\alpha(z)$ has dimension strictly smaller than δ. Applying α^{-1} and using the fact that $\alpha^{-1}(\Gamma^\circ) = \Gamma^\circ$, we have written $z \in \mathcal{G}^\circ$ as an alternating word in Γ° and $\mathcal{G}^\circ$ with more than 2 letters; a contradiction. $\square$

4   Proof of the main results

Before proving Theorem 2.1 and Corollary 2.2, we state the following lemma. It is a consequence of Lemma 8.4 in [6] (see also Props. 3.3 and 3.5 in [16]).

Lemma 4.1. *Let Γ_0, Γ_1 be ICC groups acting outerly on the II_1 factors A_0 and A_1 respectively. Set $M := A_0 \rtimes \Gamma_0$ and suppose that $\alpha : A_0 \rtimes \Gamma_0 \to A_1 \rtimes \Gamma_1$ is an isomorphism such that $\alpha(A_0) \prec_M A_1$ and $A_1 \prec_M \alpha(A_0)$.*

Then, there exists a unitary $u \in \mathcal{U}(M)$ such that $u\alpha(A_0)u^* = A_1$.

A first step in the proof of Theorem 2.1 is the following lemma. The crucial ingredients of its proof are Theorems 1.2.1 and 4.3 in [6] (see also Thms. 4.6 and 5.6 in [5] for alternative proofs).

Lemma 4.2. *Suppose that the assumptions of Theorem 2.1 are fulfilled. Set $M = M_0 *_N M_1$. For every $\alpha \in \operatorname{Aut}(M)$, there exists $u \in \mathcal{U}(M)$ such that $(\operatorname{Ad} u) \circ \alpha \in \operatorname{Aut}(N \subset M)$.*

Note that in fact assumption 3 in Theorem 2.1 will not be used in the proof of this lemma.

Proof. By 4.3 in [6] and because M_0 has property (T), there exists $i \in \{0, 1\}$ such that $\alpha(M_0) \prec_M M_i$. Since M_1 is hyperfinite, it follows that $i = 0$. So, we can take a projection $p \in M_0^n$, a non-zero partial isometry $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes M$ and a unital $*$-homomorphism $\psi : M_0 \to pM_0^n p$ satisfying $\alpha(a)v = v\psi(a)$ for all $a \in M_0$. Since $\psi(M_0)$ has property (T), we know that $\psi(M_0) \not\prec_{M_0^n} N^n$. By 1.2.1 in [6], it follows that $\psi(M_0)' \cap pM^n p \subset pM_0^n p$. In particular, $v^* v \in pM_0^n p$. So, we may assume that $p = v^* v$. Since also $\alpha(M_0)' \cap M = M_0' \cap M = \mathbb{C}1$, we have $vv^* = 1$. Factoriality of M_0 now allows to assume that $v \in \mathcal{U}(M)$ and $v^* \alpha(M_0)v \subset M_0$.

Applying the same reasoning to α^{-1}, we also get a unitary $w \in \mathcal{U}(M)$ satisfying $w^* M_0 w \subset \alpha(M_0)$. It follows that $(wv)^* M_0(wv) \subset M_0$. Another application of 1.2.1 in [6] implies that $wv \in M_0$. But then all the inclusions $(wv)^* M_0(wv) \subset v^* \alpha(M_0)v \subset M_0$ are equalities. So, after a unitary conjugacy, we may assume that $\alpha(M_0) = M_0$.

Quasi-regularity of $N \subset M$ combined with 1.2.1 in [6], implies that $\alpha(N) \prec_{M_0} N$. Similarly $N \prec_{M_0} \alpha(N)$. The lemma then follows by applying Lemma 4.1. $\qquad\square$

Proof of Theorem 2.1. We still denote $M = M_0 *_N M_1$. Let $\alpha \in \mathrm{Aut}(M)$. We prove below that after a unitary conjugacy of α, one has $\alpha(a) = a$ for all $a \in N$ and $\alpha(M_i) = M_i$ for $i \in \{0, 1\}$. This then implies that $\alpha|_{M_0} = \theta_\omega$ for some $\omega \in \mathrm{Char}(\Gamma)$ and $\alpha|_{M_1} = \sigma_g$ for some $g \in G$. Hence, it implies the surjectivity of the homomorphism $\mathrm{Char}(\Gamma) \times G \to \mathrm{Out}(M)$. The injectivity of this homomorphism follows from the irreducibility $N' \cap M = \mathbb{C}1$ that we prove now as the consequence of more general needed considerations.

Let $\mathcal{I}$ be a complete set of inequivalent irreducible unitary representations of G. For every $\pi \in \mathcal{I}$, choose a unitary $V_\pi \in \mathrm{B}(H_\pi) \otimes M_1$ satisfying $(\mathrm{id} \otimes \sigma_g)(V_\pi) = V_\pi(\pi(g) \otimes 1)$. Define $K_0(\pi) \subset M_1$ as the linear span of

$$(\xi^* \otimes a)V_\pi(\eta \otimes 1) \ , \quad \xi, \eta \in H_\pi \ , \quad a \in N \ .$$

It follows that the closure $K(\pi)$ of $K_0(\pi)$ is a finite index N-N-subbimodule of $L^2(M_1)$. Denote by

$$\Psi : \mathrm{Rep}\, G \hookrightarrow \mathrm{FAlg}(N)$$

the embedding defined in Subsection 1.4. It follows that $K(\pi) \cong (\dim \pi) \cdot \Psi(\pi)$. Moreover, we have the following orthogonal decomposition of $L^2(M_1)$.

$$L^2(M_1) = \bigoplus_{\pi \in \mathcal{I}} K(\pi) \ .$$

In the same way, we define for every $s \in \Gamma$, the subspace $H_0(s) \subset M_0$ given by $H_0(s) = Nu_s$, with closure $H(s) \subset L^2(M_0)$.

Whenever $w = s_0\pi_1 s_1 \cdots s_{n-1}\pi_n s_n$ is an alternating word in $\Gamma \setminus \{e\}$ and $\mathcal{I} \setminus \{e\}$, we define the N-N-subbimodule $H(w) \subset L^2(M)$ as the closure of $H_0(s_0)K_0(\pi_1) \cdots K_0(\pi_n)H_0(s_n)$. One then obtains the orthogonal decomposition of $L^2(M)$ given by

$$L^2(M) = \bigoplus_{w \text{ alternating word}} H(w) \ . \tag{1}$$

Moreover, as N-N-bimodules, we get the unitary equivalences

$$\begin{aligned}
H(w) &\cong H(s_0) \otimes_N K(\pi_1) \otimes_N \cdots \otimes_N K(\pi_n) \otimes_N H(s_n) \\
&\cong \text{ a multiple of } H(s_0) \otimes_N \Psi(\pi_1) \otimes_N \cdots \otimes_N \Psi(\pi_n) \otimes_N H(s_n) \ .
\end{aligned} \tag{2}$$

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Denote by $\mathrm{FAlg}(N \subset M)$ the fusion subalgebra of $\mathrm{FAlg}(N)$ generated by the finite index N-N-subbimodules of $L^2(M)$. Using assumption 3 in Theorem 2.1 and (1) and (2) above, Ψ extends to an isomorphism $\Psi : \mathbb{N}[\Gamma] * \mathrm{Rep}(G) \rightarrow \mathrm{FAlg}(N \subset M)$ such that $H(w)$ is a multiple of $\Psi(w)$ for every alternating word w. We see in particular that $L^2(M)$ contains only once the trivial N-N-bimodule (for w the empty alternating word and $H(w) = L^2(N)$). This means that $N' \cap M = \mathbb{C}1$ as was needed above.

Let $\mathrm{Char}\, G \subset \mathcal{I}$ be the subset of $\mathcal{I}$ consisting of one-dimensional unitary representations of G. Then, $\mathrm{Char}\, G$ is as well the intrinsic group of the fusion algebra $\mathrm{Rep}\, G$. Whenever $\pi \in \mathrm{Char}\, G$, we have $V_\pi \in \mathcal{U}(M_1)$ and V_π normalizes N. Whenever $w \in \Gamma * \mathrm{Char}\, G$, write $w = s_0 \pi_1 s_1 \cdots s_{n-1} \pi_n s_n$ as an alternating word in $\Gamma \setminus \{e\}$ and $\mathrm{Char}\, G \setminus \{e\}$ and define the unitary $U(w) := u_{s_0} V_{\pi_1} \cdots V_{\pi_n} u_{s_n}$ normalizing N.

We are now ready to complete the proof of the theorem. So, let $\alpha \in \mathrm{Aut}(M)$. We have to prove that after a unitary conjugacy of α, one has $\alpha(a) = a$ for all $a \in N$ and $\alpha(M_i) = M_i$ for $i \in \{0,1\}$. By Lemma 4.2, we may assume that $\alpha(N) = N$. But then, the conjugation map $K \mapsto K^\alpha$ defines an automorphism of the fusion subalgebra $\mathrm{FAlg}(N \subset M)$ of $\mathrm{FAlg}(N)$. Define the automorphism η of $\mathbb{N}[\Gamma] * \mathrm{Rep}(G)$ such that $\Psi(\eta(w)) = (\Psi(w))^\alpha$ in $\mathrm{FAlg}(N)$ for all $w \in \mathbb{N}[\Gamma] * \mathrm{Rep}(G)$. By Theorem 3.1, we find an element v in $\Gamma * \mathrm{Char}(G)$ such that $(\mathrm{Ad}\, v) \circ \eta$ globally preserves Γ and $\mathrm{Rep}(G)$.

Replacing α by $(\mathrm{Ad}\, U(v)) \circ \alpha$, we may assume that η preserves globally Γ and $\mathrm{Rep}(G)$. The equality $\alpha(N) = N$ remains true. The restrictions of η yield an automorphism of the group Γ and a permutation of $\mathcal{I}$ respecting the fusion rules. Moreover, we have $K(\pi)^\alpha \cong K(\eta(\pi))$ for every $\pi \in \mathcal{I}$ and $H(s)^\alpha \cong H(\eta(s))$ for every $s \in \Gamma$. Choose $s \in \Gamma$. Note that $H(s)^\alpha$ is isomorphic as an N-N-bimodule with the closure of $\alpha^{-1}(Nu_s)$ in ${}_N L^2(M)_N$. Since the N-N-bimodule $H(\eta(s))$ appears with multiplicity 1 in the decomposition (1) of $L^2(M)$, we conclude that the closure of $\alpha^{-1}(Nu_s)$ inside $L^2(M)$ equals $H(\eta(s))$ for all $s \in \Gamma$. It follows that $\alpha(M_0) = M_0$. A similar reasoning shows that $\alpha(M_1) = M_1$.

Since $\alpha \in \mathrm{Aut}(N \subset M_0)$, assumption 3 in Theorem 2.1 implies that the N-N-bimodule $H(\alpha|_N)$ is free with respect to $\Psi(\mathrm{Rep}\, G)$ inside $\mathrm{FAlg}(N)$. But the formula $K(\pi)^\alpha \cong K(\eta(\pi))$ means that $H(\alpha|_N)$ normalizes $\Psi(\mathrm{Rep}\, G)$. Both statements can only be true at the same time if $H(\alpha|_N)$ is the trivial N-N-module. So, $\alpha|_N$ is an inner automorphism of N and we are done. $\qquad\square$

Proof of Corollary 2.2. It suffices to give an example of an outer action of $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ on the hyperfinite II_1 factor N such that $N \rtimes \Gamma$ has property (T). Indeed, starting from a minimal action of G on the hyperfinite II_1 factor $M_1 = R$, set $N = M_1^G$ and take an outer action of Γ on N such that $M_0 := N \rtimes \Gamma$ has property (T). By Theorem 4.4 in [12], $\mathrm{Aut}(N \subset N \rtimes \Gamma)/\mathrm{Ad}\,\mathcal{U}(N)$ is a countable group. By Theorem 1.3, we can take an automorphism $\alpha \in \mathrm{Aut}(R)$ and replace $\Gamma \subset \mathrm{Aut}(N)$ by $\alpha \Gamma \alpha^{-1}$ in such a way that all conditions of Theorem 2.1 are fulfilled. Since $\mathrm{Char}\, \Gamma = \{e\}$, the corollary then follows from Theorem 2.1.

Take $\Gamma_1 := \mathrm{SL}(3, \mathbb{Z}) \ltimes (\mathbb{Z}^3 \oplus \mathbb{Z}^3)$ with the action of $\mathrm{SL}(3, \mathbb{Z})$ on $\mathbb{Z}^3 \oplus \mathbb{Z}^3$ given

$$A \cdot (x, y) := (Ax, (A^{-1})^t y) .$$

Note that Γ_1 is a property (T) group. Take $k \in \mathbb{R} \setminus 2\pi\mathbb{Q}$ and define the non degenerate 2-cocycle $\Omega \in Z^2(\mathbb{Z}^3 \oplus \mathbb{Z}^3, S^1)$ by the formula

$$\Omega\big((x, y); (x', y')\big) := e^{\mathbf{i}k(\langle x, y'\rangle - \langle y, x'\rangle)}$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{Z}^3$. The 2-cocycle Ω is $\mathrm{SL}(3, \mathbb{Z})$-invariant and hence extends to a 2-cocycle $\tilde{\Omega} \in Z^2(\Gamma_1, S^1)$ by the formula

$$\tilde{\Omega}\big(((x, y), A); ((x', y'), B)\big) := \Omega\big((x, y); A \cdot (x', y')\big) .$$

The twisted group von Neumann algebra $\mathcal{L}_{\tilde{\Omega}}(\Gamma_1)$ still has property (T) and can be regarded as well as $\mathcal{L}_\Omega(\mathbb{Z}^3 \oplus \mathbb{Z}^3) \rtimes \mathrm{SL}(3, \mathbb{Z})$. Since $\mathcal{L}_\Omega(\mathbb{Z}^3 \oplus \mathbb{Z}^3)$ is the hyperfinite II_1 factor, we are done. $\qquad\square$

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